"HENRY COANDA"
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ROMANIA

GERMANY

" GENERAL M.R. STEFANIK"
ARMED FORCES ACADEMY
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A SPLITTING ALGORITHM FOR VARIATIONAL INEQUALITIES

Corina L. CHIRIAC

Bioterra University, Bucharest, Romania

Abstract: *Let X be a Banach space, X^* its dual,$(\cdot,\cdot)$ the duality on $X^*\times X$, and $T:X\to X^*$ an operator (nonlinear and multi-valued, in general), and $\varphi:X\to \mathbf{R}\cup\{+\infty\}$ is a convex lower semi-continuous function with the effective domain $D(\varphi)=\{v\in X \mid \varphi(v)<\infty$. Given a closed convex subset $K\subset X$ and $f\in X^*$, the problem of finding $u\in K$ such that $(Tu-f, v-u)+\varphi(v)-\varphi(u)\geq 0$ for all $v\in D(\varphi)$ is called a variational inequality.In this paper we applied a splitting algorithm for the stationary inclusion $Tu+\partial\varphi(u)\ni 0$, equivalent with a variational inequality, when T is a maximal monotone mapping on a Hilbert space H. Also we establish conditions to prove the weak convergence of the algorithm..*

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1. INTRODUCTION

A fundamental algorithm for finding roots of a monotone operator is the *proximal point algorithm* (see [12]). This method requires evaluation of *resolvent* operators of the form $(I+\lambda S)^{-1}$, where S is monotone and set-valued, λ is a positive scalar, and I denotes the identity mapping. The main difficulty with the method is that $I+\lambda S$ may be hard to invert, depending on the nature of S. One alternative is to find maximal monotone operators W and V such that W+V=S, but $I+\lambda W$ and $I+\lambda V$ are easier to invert than $I+\lambda S$. One can then devise an algorithm that uses only operators of the form $(I+\lambda W)^{-1}$ and $(I+\lambda V)^{-1}$, rather than $(I+\lambda(W+V))^{-1}$. Such an approach is called a *splitting method***,** and is inspired by well-established techniques from numerical linear algebra (see, for example [10]).

A number of authors have extensively studied monotone operator, splitting methods, which fall into four principal classes: forward-backward [13], double-backward [8], Peaceman-Rachford [9], and Douglas-Rachford [9].

We will focus on the Peaceman-Rachford algorithm,for variational inequalities in the case of multi-valued monotone operators. We will prove the convergence of this algorithm.

2. SPLITTING ALGORITHMS FOR STATIONARY PROBLEMS

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $S:H\to 2^H$ be a monotone operator. We study the nonlinear multi-valued stationary equation $Su\ni 0$.

We consider the case when S=W+V and W,V are maximal monotone. For that we get

V,W single-valued operators. In conclusion we have to solve the inclusion

$$Wu+Vu \ni 0 \quad (1)$$

equivalent with

$$u+\lambda Wu \ni (u-\lambda Vu),$$

where $\lambda>0$ is a constant. Since W is a maximal monotone operator, we deduce that

$$u=(I+\lambda W)^{-1}(I-\lambda V)u. \quad (2)$$

Analogously, (1) can be written as

$$u+\lambda Vu \ni (u-\lambda Wu),$$

where $\lambda>0$ is a constant, and finally we obtain

$$u=(I+\lambda V)^{-1}(I-\lambda W)u. \quad (3)$$

Combining (2) and (3) we have

$$u=(I+\lambda V)^{-1}(I-\lambda W)(I+\lambda W)^{-1}(I-\lambda V)u. \quad (4)$$

Relation (4) suggests the following algorithm

$$u^{n+1}=(I+\lambda V)^{-1}(I-\lambda W)(I+\lambda W)^{-1}(I-\lambda V)u^n \quad (5)$$

which was introduced, in the case of linear operators, by Peaceman-Rachford (see [11]).

When V is single-valued, we have the identity

$$(I+\lambda V)(I+\lambda V)^{-1}=I. \quad (6)$$

From (2) and (6), we obtain

$$u=(I+\lambda V)^{-1}[(I+\lambda W)^{-1}(I-\lambda V)+\lambda V]u. \quad (7)$$

Relation (7) suggests the iterative scheme

$$u^{n+1}=(I+\lambda V)^{-1}[(I+\lambda W)^{-1}(I-\lambda V)+\lambda V]u^n \quad (8)$$

which was introduced by Douglas-Rachford [6].

These algorithms are both unconditionally stable (u^n remains bounded independently of n for any λ). This set of properties is remarkable if we compare them to what we get with more standard algorithms.

The first one is

$$u^{n+1}=(I+\lambda W)^{-1}(I-\lambda V)u^n, \quad (9)$$

which is not unconditionally stable, but converges to the solution of the stationary problem for λ sufficiently small if V is Lipschitz continuous (see [7], [4]).

The second one is

$$u^{n+1}=(I+\lambda W)^{-1}(I+\lambda V)^{-1}u^n,$$

which is unconditionally stable but does not converge to the solution of the stationary problem for any λ, except with some special modification (see Lions [8]).

All these are called *splitting algorithms* since, up to the introduction of a fractionary step, they can be interpreted as the combination of a step for W and a step for V. As an example, (9) can be written

$$\frac{1}{\lambda}(u^{n+1/2}-u^n)+Vu^n=0,$$

$$\frac{1}{\lambda}(u^{n+1}-u^{n+1/2})+Wu^{n+1}=0,$$

which shows that (9) results from the combination of a forward step on V and backward step on W. In this section, we show that these algorithms can be used to solve variational inequalities

We shall assume that $T:H\to 2^H$ is a maximal monotone operator. We denote by D(T) the domain of T and by

$$J_T^\lambda=(I+\lambda T)^{-1}$$

the resolvent of T. Let $\varphi:H\to \mathbf{R}\cup\{+\infty\}$ be a proper convex lower semi-continuous (l.s.c) function and let f be defined on H. We consider the following variational inequality:

Find $u\in D(\varphi)$ such that there exists $w\in Tu$ satisfying

$$(w-f,v-u)+\varphi(v)-\varphi(u)\geq 0 \quad \forall\, v\in D(\varphi). \quad (10)$$

It is easy to notice that the above problem is equivalent with the problem:

Find $u\in H$ such that

$$Tu+\partial\varphi(u)\ni f \quad (11)$$

where $\partial\varphi$ is the subdifferential of φ.

We shall assume that the problem (10) has at least one solution. Hence there exists $u\in H$, $t\in Tu$, $s\in\partial\varphi(u)$ such that t+s=f. We do not affect the generalization of the problem if we assume in the sequel that $f\equiv 0$.

Since T and $\partial\varphi$ are maximal monotone operators, we can apply the Peaceman-Rachford algorithm to solve the problem:

Find $u\in H$ such that

$$Tu+\partial\varphi(u)\ni 0. \quad (12)$$

In the case considered here, where T and $\partial\varphi$ are multi-valued, we need to make precise the definition of the algorithms (5) and (8). For both, $u^0\in D(T)$ is given, and we choose $t^0\in Tu^0$ and set $v^0=u^0+\lambda t^0$ in such a way that $u^0=J_T^\lambda(v^0)$. We then define by induction the sequence $\{v^n\}$ in the following way:

Algorithm

$$v^{n+1}=(2J_{\partial\varphi}^\lambda-I)(2J_T^\lambda-I)v^n \quad (13)$$

Convergence of the Algorithm . We obtain the algorithm above after a simple computation. Since t+s=0 and $v=u+\lambda t$ in such way that $u=J_T^\lambda(v)$, we have

$s=-t$, $u+\lambda s=u-\lambda t=2u-u-\lambda t=2J_T^\lambda(v)-v$.

From this relation, we obtain

$$u=J_{\partial\varphi}^\lambda(2J_T^\lambda-I)v,$$

$$2u=2J_{\partial\varphi}^\lambda(2J_T^\lambda-I)v,$$

$$v+2J_T^\lambda(v)-v=2J_{\partial\varphi}^\lambda(2J_T^\lambda-I)v,$$

$$v=(2J_{\partial\varphi}^\lambda-I)(2J_T^\lambda-I)v,$$

relation which suggests our algorithm, and the following notations

$$v=u+\lambda t,\ w=u+\lambda s,$$

$$w^n=2u^n-v^n,\ t^n=\frac{1}{\lambda}(v^n-u^n),\ s^n=\frac{1}{2\lambda}(w^n-v^{n+1}).$$

We prove the following result.

Proposition 2.1. *Under the assumption: there exists* $u\in H$, $t\in Tu$, $s\in\partial\varphi(u)$ *such that* $t+s=0$, *the sequences* $\{u^n\}$, $\{v^n\}$, $\{w^n\}$, $\{s^n\}$, $\{t^n\}$, *remain bounded. Moreover*

$$\lim_{n\to\infty}(t^n-t,\ u^n-u)=0 \qquad (14)$$

$$\lim_{n\to\infty}\left(s^n-s,\ \frac{v^{n+1}+w^n}{2}-u\right)=0. \qquad (15)$$

Proof. From the definition of t^n, we have $v^n=u^n+\lambda t^n$. As $u^n=J_T^\lambda v^n$, we have $v^n\in u^n+\lambda Tu^n$, hence $t^n\in Tu^n$. From the monotonicity of T, we get:

$$0\le(t^n-t,u^n-u)=\frac{1}{4\lambda}(\|v^n-v\|^2-\|w^n-w\|^2), \quad (16)$$

using the relations

$$u^n=\frac{1}{2}(v^n+w^n),\ u=\frac{1}{2}(v+w),$$

$$t^n=\frac{1}{2\lambda}(v^n-w^n),\ t=\frac{1}{2\lambda}(v-w).$$

On the other hand, from (13), we have

$$v^{n+1}=(2J_{\partial\varphi}^\lambda-I)w^n\Rightarrow\frac{1}{2}(v^{n+1}+w^n)=J_{\partial\varphi}^\lambda(w^n).$$

Hence

$$w^n\in\frac{v^{n+1}+w^n}{2}+\lambda\partial\varphi\left(\frac{v^{n+1}+w^n}{2}\right),$$

$$s^n=\frac{w^n-v^{n+1}}{2\lambda}\in\partial\varphi\left(\frac{v^{n+1}+w^n}{2}\right).$$

From the monotonicity of $\partial\varphi$ we deduce

$$0\le\left(s^n-s,\ \frac{v^{n+1}+w^n}{2}-u\right)=$$

$$\left(\frac{w^n-v^{n+1}}{2\lambda}-\frac{w-v}{2\lambda},\ \frac{w^n+v^{n+1}}{2}-\frac{w+v}{2}\right)=$$

$$\frac{1}{4\lambda}(\|w^n-w\|^2-\|v^{n+1}-v\|^2). \qquad (17)$$

From (16) and (17) we obtain the inequalities

$$\|v^{n+1}-v\|^2\le\|w^n-w\|^2\le\|v^n-v\|^2,$$

which show that the sequences $\{v^n\},\{w^n\}$ are bounded. Implicitly, $\{u^n\}$ is bounded. Finally, as

$$\|v^n-v\|^2-\|v^{n+1}-v\|^2\to 0 \text{ as } n\to\infty,$$

(16), (17) imply (14) and (15). ■

Definition 2.2. We say that $A:H\to H$ satisfies *condition (C)* if for all x^n, $\bar{x}\in D(A)$ such that Ax^n is bounded, $x^n\rightharpoonup\bar{x}$ (weak convergence), and $\overline{(Ax^n-Ax,\ x^n-x)}\to 0$ as $n\to+\infty$, imply $x=\bar{x}$.

Theorem 2.3. *If* T *is single-valued and satisfies condition (C), then the sequence* $\{u^n\}$ *obtained from Algorithm converges weakly to* u, *the solution of (12), which is unique.*

Proof. We first prove uniqueness. Let u_1, u_2, be two solutions of (12). We have, using the monotonicity of $\partial\varphi$,

$$0\le(Tu_1-Tu_2,\ u_1-u_2)=$$

$$-(\partial\varphi(u_1)-\partial\varphi(u_2),\ u_1-u_2)\le 0,$$

hence $(Tu_1-Tu_2,\ u_1-u_2)=0$ which, together with condition (C) implies $u_1=u_2$.

Let $\{u^{n_i}\}$ be a subsequence of the bounded sequence $\{u^n\}$ such that $u^{n_i}\rightharpoonup\bar{u}$. From (14) and condition (C), one gets $u=\bar{u}$, and from the uniqueness, the whole sequence $\{u^n\}$ converges weakly to u. ■

Remark 2.4. We can prove that, if a subsequence $\{v^{n_i}\}$ of $\{v^n\}$ is bounded, then

the problem (12) has one solution u. Indeed, let $A=(2J^{\lambda}_{\partial\varphi}-I)(2J^{\lambda}_{T}-I)$, we have $v^{n+1}=Av^{n}$.As $2J^{\lambda}_{\partial\varphi}-I$ and $2J^{\lambda}_{T}-I$ are nonexpansive, A itself is nonexpansive. Because the subsequence $\{v^{n_i}\}$ is bounded, we obtain that A has a fixed point v, with Av=v. Let $u=J^{\lambda}_{T}v$ we have $u\in D(T)$ and $v=(2J^{\lambda}_{\partial\varphi}-I)(2u-v)$, $u=J^{\lambda}_{\partial\varphi}(2u-v)$.
Hence $u\in D(\partial\varphi)$. Let $t\in Tu$ satisfy $v=u+\lambda t$. We have $u=J^{\lambda}_{\partial\varphi}(u-\lambda t) \Rightarrow (u-\lambda t)\in u+\lambda\partial\varphi(u)$ that is $-t\in\partial\varphi(u)$, hence u is a solution of the problem (12).

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